

# DYNAMICS OF TWO-DIMENSIONAL EVOLUTION ALGEBRAS

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**ABSTRACT.** Recently in [2] a notion of a chain of evolution algebras is introduced. This chain is a dynamical system the state of which at each given time is an evolution algebra. The sequence of matrices of the structural constants for this chain of evolution algebras satisfies the Chapman-Kolmogorov equation. In this paper we construct 25 distinct examples of chains of two-dimensional evolution algebras. For all of these 25 chains we study the behavior of the baric property, the behavior of the set of absolute nilpotent elements and dynamics of the set of idempotent elements depending on the time.

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## 1. INTRODUCTION

The concept of a dynamical system has its origins in Newtonian mechanics. There, as in other natural sciences and engineering disciplines, the evolution rule of a dynamical systems is given implicitly by a relation that gives the state of the system only a short time into the future. The relation is either a differential equation, a difference equation or another time scale. To determine the state for all future times requires iterating the relation many times each advancing time by a small step.

For simple dynamical systems, knowing the trajectory is often sufficient, but most dynamical systems are too complicated to be understood in terms of individual trajectories.

Following Kolmogorov, an approach based on the time evolution of initial distributions appears to be more appropriate (see e.g. [1]).

In [9] a notion of evolution algebra is introduced. This evolution algebra is defined as follows. Let  $(E, \cdot)$  be an algebra over a field  $K$ . If it admits a basis  $e_1, e_2, \dots$ , such that  $e_i \cdot e_j = 0$ , if  $i \neq j$  and  $e_i \cdot e_i = \sum_k a_{ik} e_k$ , for any  $i$ , then this algebra is called an *evolution algebra*.

The concept of evolution algebras lies between algebras and dynamical systems. Algebraically, evolution algebras are non-associative Banach algebra; dynamically, they represent discrete dynamical systems. Evolution algebras have many connections with other mathematical fields including graph theory, group theory, stochastic processes, mathematical physics, etc.

In the book [9], the foundation of evolution algebra theory and applications in non-Mendelian genetics and Markov chains are developed.

In [8] the algebraic structures of function spaces defined by graphs and state spaces equipped with Gibbs measures by associating evolution algebras are studied. Results of [8] also allow a natural introduction of thermodynamics in studying of several systems of biology, physics and mathematics by theory of evolution algebras.

There exist several classes of non-associative algebras (baric, evolution, Bernstein, train, stochastic, etc.), whose investigation has provided a number of significant contributions to theoretical population genetics. Such classes have been defined different times by several authors, and all algebras belonging to these classes are generally called "genetic" [3–5], [6, 7, 9, 10].

Recently in [2] a notion of a chain of evolution algebras is introduced. This chain is a dynamical system the state of which at each given time is an evolution algebra. The chain is defined by the sequence of matrices of the structural constants (of the evolution algebras considered in [9]) which satisfies the Chapman-Kolmogorov equation. In [2] several examples (time homogenous, time non-homogenous, periodic, etc.) of such chains are given.

In this paper we continue investigation of chain of evolution algebras, we in more detail study chains generated by two-dimensional evolution algebras.

The paper is organized as follows. In Section 2 we give main definitions and facts related to a chain of evolution algebras. In Section 3 we construct 25 distinct chains of two-dimensional evolution algebras. In Section 4 we study behavior of the property to be baric for each 25 chains constructed in the section 3. We show that some of the chains are baric any time, some of them are never baric. For other chains of 25 ones which have (baric property transition) we define a baric property controller function and under some conditions on this controller we prove that the chain is not baric almost surely (with respect to Lebesgue measure). For each 25 chains of evolution algebras in Section 5 we study the behavior of the set of absolute nilpotent elements and in the last section we study dynamics of the set of idempotent elements depending on the time.

## 2. PRELIMINARIES

Following [2] we consider a family  $\{E^{[s,t]} : s, t \in \mathbb{R}, 0 \leq s \leq t\}$  of  $n$ -dimensional evolution algebras over the field  $\mathbb{R}$ , with basis  $e_1, \dots, e_n$  and multiplication table

$$(2.1) \quad e_i e_i = \sum_{j=1}^n a_{ij}^{[s,t]} e_j, \quad i = 1, \dots, n; \quad e_i e_j = 0, \quad i \neq j.$$

Here parameters  $s, t$  are considered as time.

Denote by  $\mathcal{M}^{[s,t]} = \left( a_{ij}^{[s,t]} \right)_{i,j=1,\dots,n}$  -the matrix of structural constants.

**Definition 2.1.** [2] A family  $\{E^{[s,t]} : s, t \in \mathbb{R}, 0 \leq s \leq t\}$  of  $n$ -dimensional evolution algebras over the field  $\mathbb{R}$  is called a chain of evolution algebras (CEA) if the matrix  $\mathcal{M}^{[s,t]}$  of structural constants satisfies the Chapman-Kolmogorov equation

$$(2.2) \quad \mathcal{M}^{[s,t]} = \mathcal{M}^{[s,\tau]} \mathcal{M}^{[\tau,t]}, \quad \text{for any } s < \tau < t.$$

**Definition 2.2.** A CEA is called a time-homogenous CEA if the matrix  $\mathcal{M}^{[s,t]}$  depends only on  $t - s$ . In this case we write  $\mathcal{M}^{[t-s]}$ .

**Definition 2.3.** A CEA is called periodic if its matrix  $\mathcal{M}^{[s,t]}$  is periodic with respect to at least one of the variables  $s, t$ , i.e. (periodicity with respect to  $t$ )  $\mathcal{M}^{[s,t+P]} = \mathcal{M}^{[s,t]}$  for all values of  $t$ . The constant  $P$  is called the period, and is required to be nonzero.

Let  $\{\mathcal{M}^{[s,t]}, 0 \leq s \leq t\}$  be a family of stochastic matrices which satisfies the equation (2.2), then it defines a Markov process. Thus we have

**Theorem 2.4.** [2] For each Markov process, there is a CEA whose structural constants are transition probabilities of the process, and whose generator set (basis) is the state space of the Markov process.

A character for an algebra  $A$  is a nonzero multiplicative linear form on  $A$ , that is, a nonzero algebra homomorphism from  $A$  to  $\mathbb{R}$  [6]. Not every algebra admits a character. For example, an algebra with the zero multiplication has no character.

**Definition 2.5.** A pair  $(A, \sigma)$  consisting of an algebra  $A$  and a character  $\sigma$  on  $A$  is called a baric algebra. The homomorphism  $\sigma$  is called the weight (or baric) function of  $A$  and  $\sigma(x)$  the weight (baric value) of  $x$ .

In [6] for the evolution algebra of a free population it is proven that there is a character  $\sigma(x) = \sum_i x_i$ , therefore that algebra is baric. But the evolution algebra  $E$  introduced in [9] is not baric, in general. The following theorem gives a criterion for an evolution algebra  $E$  to be baric.

**Theorem 2.6.** [2] An  $n$ -dimensional evolution algebra  $E$ , over the field  $\mathbb{R}$ , is baric if and only if there is a column  $(a_{1i_0}, \dots, a_{ni_0})^T$  of its structural constants matrix  $\mathcal{M} = (a_{ij})_{i,j=1,\dots,n}$ , such that  $a_{i_0i_0} \neq 0$  and  $a_{ii_0} = 0$ , for all  $i \neq i_0$ . Moreover, the corresponding weight function is  $\sigma(x) = a_{i_0i_0}x_{i_0}$ .

In [2] several concrete examples of CEAs are given and their time-dynamics are studied. In this paper we continue investigation of CEAs, in more detail we study CEAs generated by two-dimensional evolution algebras.

### 3. CONSTRUCTION OF CHAINS OF TWO-DIMENSIONAL EAS

In this section we shall construct several chains of two-dimensional evolution algebras, which were not considered in [2].

To construct a chain of two-dimensional evolution algebra one has to solve equation (2.2) for  $2 \times 2$  matrix  $\mathcal{M}^{[s,t]}$ . This equation gives the following system of functional equations (with four unknown functions):

$$(3.1) \quad \begin{aligned} a_{11}^{[s,t]} &= a_{11}^{[s,\tau]} a_{11}^{[\tau,t]} + a_{12}^{[s,\tau]} a_{21}^{[\tau,t]}, \\ a_{12}^{[s,t]} &= a_{11}^{[s,\tau]} a_{12}^{[\tau,t]} + a_{12}^{[s,\tau]} a_{22}^{[\tau,t]}, \\ a_{21}^{[s,t]} &= a_{21}^{[s,\tau]} a_{11}^{[\tau,t]} + a_{22}^{[s,\tau]} a_{21}^{[\tau,t]}, \\ a_{22}^{[s,t]} &= a_{21}^{[s,\tau]} a_{12}^{[\tau,t]} + a_{22}^{[s,\tau]} a_{22}^{[\tau,t]}. \end{aligned}$$

But the analysis of the system (3.1) is difficult. We shall consider several cases where the system is solvable:

**Case 1.**  $a_{11}^{[s,t]} = a_{22}^{[s,t]} = \alpha(s, t)$ ,  $a_{12}^{[s,t]} = a_{21}^{[s,t]} = \beta(s, t)$ . Then equation (3.1) is reduced to

$$\begin{aligned} \alpha(s, t) &= \alpha(s, \tau) \alpha(\tau, t) + \beta(s, \tau) \beta(\tau, t); \\ \beta(s, t) &= \alpha(s, \tau) \beta(\tau, t) + \beta(s, \tau) \alpha(\tau, t). \end{aligned}$$

Denote

$$(3.2) \quad f(s, t) = \alpha(s, t) + \beta(s, t), \quad \varphi(s, t) = \alpha(s, t) - \beta(s, t),$$

then the last system of functional equations can be written as

$$f(s, t) = f(s, \tau) f(\tau, t), \quad \varphi(s, t) = \varphi(s, \tau) \varphi(\tau, t), \quad s \leq \tau \leq t.$$

Both these equations are known as Cantor's second equation which has very rich family of solutions:

- a)  $f(s, t) \equiv 0$ ;
- b)  $f(s, t) = \frac{\Phi(t)}{\Phi(s)}$ , where  $\Phi$  is an arbitrary function with  $\Phi(s) \neq 0$ ;
- c)

$$f(s, t) = \begin{cases} 1, & \text{if } s \leq t < a, \\ 0, & \text{if } t \geq a. \end{cases} \quad \text{where } a > 0.$$

Similarly,

- a')  $\varphi(s, t) \equiv 0$ ;
- b')  $\varphi(s, t) = \frac{\Psi(t)}{\Psi(s)}$ , where  $\Psi$  is an arbitrary function with  $\Psi(s) \neq 0$ ;
- c')

$$\varphi(s, t) = \begin{cases} 1, & \text{if } s \leq t < b, \\ 0, & \text{if } t \geq b. \end{cases} \quad \text{where } b > 0.$$

Substituting these solutions into (3.2) and finding  $\alpha(s, t)$  and  $\beta(s, t)$  we get the following matrices

$$\mathcal{M}_0^{[s,t]} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad \mathcal{M}_1^{[s,t]} = \frac{1}{2} \begin{pmatrix} \frac{\Psi(t)}{\Psi(s)} & -\frac{\Psi(t)}{\Psi(s)} \\ -\frac{\Psi(t)}{\Psi(s)} & \frac{\Psi(t)}{\Psi(s)} \end{pmatrix};$$

$$\begin{aligned}
\mathcal{M}_2^{[s,t]} &= \frac{1}{2} \begin{cases} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, & \text{if } s \leq t < b; \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } t \geq b \end{cases}; \\
\mathcal{M}_3^{[s,t]} &= \frac{1}{2} \begin{pmatrix} \frac{\Phi(t)}{\Phi(s)} & \frac{\Phi(t)}{\Phi(s)} \\ \frac{\Phi(t)}{\Phi(s)} & \frac{\Phi(t)}{\Phi(s)} \end{pmatrix}. \\
\mathcal{M}_4^{[s,t]} &= \frac{1}{2} \begin{pmatrix} \frac{\Phi(t)}{\Phi(s)} + \frac{\Psi(t)}{\Psi(s)} & \frac{\Phi(t)}{\Phi(s)} - \frac{\Psi(t)}{\Psi(s)} \\ \frac{\Phi(t)}{\Phi(s)} - \frac{\Psi(t)}{\Psi(s)} & \frac{\Phi(t)}{\Phi(s)} + \frac{\Psi(t)}{\Psi(s)} \end{pmatrix}. \\
\mathcal{M}_5^{[s,t]} &= \frac{1}{2} \begin{cases} \begin{pmatrix} \frac{\Phi(t)}{\Phi(s)} + 1 & \frac{\Phi(t)}{\Phi(s)} - 1 \\ \frac{\Phi(t)}{\Phi(s)} - 1 & \frac{\Phi(t)}{\Phi(s)} + 1 \end{pmatrix}, & \text{if } s \leq t < b; \\ \begin{pmatrix} \frac{\Phi(t)}{\Phi(s)} & \frac{\Phi(t)}{\Phi(s)} \\ \frac{\Phi(t)}{\Phi(s)} & \frac{\Phi(t)}{\Phi(s)} \end{pmatrix}, & \text{if } t \geq b \end{cases}; \\
\mathcal{M}_6^{[s,t]} &= \frac{1}{2} \begin{cases} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{if } s \leq t < a; \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } t \geq a \end{cases}; \\
\mathcal{M}_7^{[s,t]} &= \frac{1}{2} \begin{cases} \begin{pmatrix} 1 + \frac{\Psi(t)}{\Psi(s)} & 1 - \frac{\Psi(t)}{\Psi(s)} \\ 1 - \frac{\Psi(t)}{\Psi(s)} & 1 + \frac{\Psi(t)}{\Psi(s)} \end{pmatrix}, & \text{if } s \leq t < a; \\ \begin{pmatrix} \frac{\Psi(t)}{\Psi(s)} & -\frac{\Psi(t)}{\Psi(s)} \\ -\frac{\Psi(t)}{\Psi(s)} & \frac{\Psi(t)}{\Psi(s)} \end{pmatrix}, & \text{if } t \geq a \end{cases}; \\
\mathcal{M}_8^{[s,t]} &= \frac{1}{2} \begin{cases} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, & \text{if } s \leq t < \min\{a, b\}; \\ \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, & \text{if } a \leq t < b, a < b; \\ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \text{if } b \leq t < a, a > b; \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } t \geq \max\{a, b\} \end{cases}.
\end{aligned}$$

Thus in this case we have nine CEAs:  $E_i^{[s,t]}$ ,  $0 \leq s \leq t$  which correspond to  $\mathcal{M}_i^{[s,t]}$ ,  $i = 0, 1, \dots, 8$  listed above.

**Case 2.**  $a_{11}^{[s,t]} = a_{22}^{[s,t]} = \alpha(s, t)$ ,  $a_{12}^{[s,t]} = -a_{21}^{[s,t]} = \beta(s, t)$ . Then equation (3.1) is reduced to

$$(3.3) \quad \begin{aligned} \alpha(s, t) &= \alpha(s, \tau)\alpha(\tau, t) - \beta(s, \tau)\beta(\tau, t); \\ \beta(s, t) &= \alpha(s, \tau)\beta(\tau, t) + \beta(s, \tau)\alpha(\tau, t). \end{aligned}$$

It is easy to check that this system of functional equations has a solution

$$\alpha(s, t) = \cos(t - s), \quad \beta(s, t) = \sin(t - s).$$

This gives the following matrix

$$(3.4) \quad \mathcal{M}_9^{[s,t]} = \begin{pmatrix} \cos(t - s) & \sin(t - s) \\ -\sin(t - s) & \cos(t - s) \end{pmatrix}.$$

Note that this matrix defines a periodic CEA. But we do not know another solution (with  $\alpha\beta \neq 0$ ) of the system (3.3).

We denote by  $E_9^{[s,t]} = E_9^{[t-s]}$  the CEA which corresponds to  $\mathcal{M}_9^{[s,t]}$ .

**Case 3.**  $a_{11}^{[s,t]} = a_{21}^{[s,t]} = \alpha(s, t)$ ,  $a_{12}^{[s,t]} = a_{22}^{[s,t]} = \beta(s, t)$ . In this case the equation (3.1) is reduced to

$$\begin{aligned} \alpha(s, t) &= \alpha(\tau, t)(\alpha(s, \tau) + \beta(s, \tau)); \\ \beta(s, t) &= \beta(\tau, t)(\alpha(s, \tau) + \beta(s, \tau)). \end{aligned}$$

Denote

$$(3.5) \quad \gamma(s, t) = \alpha(s, t) + \beta(s, t), \quad \delta(s, t) = \alpha(s, t) - \beta(s, t),$$

then the last system of functional equations can be written as

$$\gamma(s, t) = \gamma(s, \tau)\gamma(\tau, t), \quad \delta(s, t) = \gamma(s, \tau)\delta(\tau, t), \quad s \leq \tau \leq t.$$

The first equation is Cantor's second equation which has solutions:

- a)  $\gamma(s, t) \equiv 0$ ;
- b)  $\gamma(s, t) = \frac{h(t)}{h(s)}$ , where  $h$  is an arbitrary function with  $h(s) \neq 0$ ;
- c)

$$\gamma(s, t) = \begin{cases} 1, & \text{if } s \leq t < a, \\ 0, & \text{if } t \geq a. \end{cases} \quad \text{where } a > 0.$$

Using these solution from the second equation we find  $\delta$ :

- a')  $\delta(s, t) \equiv 0$ ;
- b')  $\delta(s, t) = \frac{g(t)}{h(s)}$ , where  $g$  is an arbitrary function;
- c')

$$\delta(s, t) = \begin{cases} \psi(t), & \text{if } s \leq t < a, \\ 0, & \text{if } t \geq a. \end{cases} \quad \text{where } a > 0, \psi(t) \text{ is an arbitrary function.}$$

Substituting these solutions into (3.5) we get the following (non-zero) matrices

$$\mathcal{M}_{10}^{[s,t]} = \frac{1}{2} \begin{pmatrix} \frac{h(t)+g(t)}{h(s)} & \frac{h(t)-g(t)}{h(s)} \\ \frac{h(t)+g(t)}{h(s)} & \frac{h(t)-g(t)}{h(s)} \end{pmatrix}.$$

$$\mathcal{M}_{11}^{[s,t]} = \frac{1}{2} \begin{cases} \begin{pmatrix} 1+\psi(t) & 1-\psi(t) \\ 1+\psi(t) & 1-\psi(t) \end{pmatrix}, & \text{if } s \leq t < a; \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } t \geq a \end{cases}.$$

Thus in this case we have two new CEAs:  $E_i^{[s,t]}$ ,  $0 \leq s \leq t$  which correspond to  $\mathcal{M}_i^{[s,t]}$ ,  $i = 10, 11$  listed above.

**Case 4.**  $a_{11}^{[s,t]} = a_{12}^{[s,t]} = \alpha(s, t)$ ,  $a_{21}^{[s,t]} = a_{22}^{[s,t]} = \beta(s, t)$ . In this case the equation (3.1) is reduced to

$$\alpha(s, t) = \alpha(s, \tau)(\alpha(\tau, t) + \beta(\tau, t));$$

$$\beta(s, t) = \beta(s, \tau)(\alpha(\tau, t) + \beta(\tau, t)).$$

Denote

$$(3.6) \quad \gamma(s, t) = \alpha(s, t) + \beta(s, t), \quad \delta(s, t) = \alpha(s, t) - \beta(s, t),$$

then the last system of functional equations can be written as

$$\gamma(s, t) = \gamma(s, \tau)\gamma(\tau, t), \quad \delta(s, t) = \delta(s, \tau)\gamma(\tau, t), \quad s \leq \tau \leq t.$$

The first equation has solutions:

- a)  $\gamma(s, t) \equiv 0$ ;
- b)  $\gamma(s, t) = \frac{h(t)}{h(s)}$ , where  $h$  is an arbitrary function with  $h(s) \neq 0$ ;
- c)

$$\gamma(s, t) = \begin{cases} 1, & \text{if } s \leq t < a, \\ 0, & \text{if } t \geq a. \end{cases} \quad \text{where } a > 0.$$

Using these solution from the second equation we find  $\delta$ :

- a')  $\delta(s, t) \equiv 0$ ;
- b')  $\delta(s, t) = g(s)h(t)$ , where  $g$  is an arbitrary function;
- c')

$$\delta(s, t) = \begin{cases} \psi(s), & \text{if } s \leq t < a, \\ 0, & \text{if } t \geq a. \end{cases} \quad \text{where } a > 0, \psi(t) \text{ is an arbitrary function.}$$

Consequently, we get the following (non-zero) new matrices

$$\mathcal{M}_{12}^{[s,t]} = \frac{1}{2} \begin{pmatrix} h(t) \left( \frac{1}{h(s)} + g(s) \right) & h(t) \left( \frac{1}{h(s)} + g(s) \right) \\ h(t) \left( \frac{1}{h(s)} - g(s) \right) & h(t) \left( \frac{1}{h(s)} - g(s) \right) \end{pmatrix};$$

$$\mathcal{M}_{13}^{[s,t]} = \frac{1}{2} \begin{cases} \begin{pmatrix} 1 + \psi(s) & 1 + \psi(s) \\ 1 - \psi(s) & 1 - \psi(s) \end{pmatrix}, & \text{if } s \leq t < a; \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } t \geq a \end{cases}.$$

Hence in this case we have two new CEAs:  $E_i^{[s,t]}$ ,  $0 \leq s \leq t$  which correspond to  $\mathcal{M}_i^{[s,t]}$ ,  $i = 12, 13$  listed above.

**Case 5.**  $a_{12}^{[s,t]} \equiv 0$ , (the case  $a_{21}^{[s,t]} \equiv 0$  is similar) then the system (3.1) reduced to following

$$(3.7) \quad \begin{aligned} a_{11}^{[s,t]} &= a_{11}^{[s,\tau]} a_{11}^{[\tau,t]}, \\ a_{21}^{[s,t]} &= a_{21}^{[s,\tau]} a_{11}^{[\tau,t]} + a_{22}^{[s,\tau]} a_{21}^{[\tau,t]}, \\ a_{22}^{[s,t]} &= a_{22}^{[s,\tau]} a_{22}^{[\tau,t]} \end{aligned}$$

The first and the third equations of the system (3.7) are Cantor's second equations. Substituting solutions of these equations into the second equation of the system (3.7) we find function  $a_{21}^{[s,t]}$ . Note that in many cases the second equation of the system (3.7) will be reduced to Cantor's first equation:

$$\gamma(s, t) = \gamma(s, \tau) + \gamma(\tau, t),$$

which also has very rich family of solutions:  $\gamma(s, t) = \Psi(t) - \Psi(s)$ , where  $\Psi$  is an arbitrary function. Thus solving the system (3.7) we obtain the following new matrices:

$$\begin{aligned} \mathcal{M}_{14}^{[s,t]} &= \begin{pmatrix} \frac{\Phi(t)}{\Phi(s)} & 0 \\ \Phi(t)\psi(s) & 0 \end{pmatrix}; \\ \mathcal{M}_{15}^{[s,t]} &= \begin{cases} \begin{pmatrix} 1 & 0 \\ \psi(s) & 0 \end{pmatrix}, & \text{if } s \leq t < a; \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } t \geq a \end{cases}; \\ \mathcal{M}_{16}^{[s,t]} &= \begin{pmatrix} 0 & 0 \\ \frac{g(t)}{\psi(s)} & \frac{\psi(t)}{\psi(s)} \end{pmatrix}. \\ \mathcal{M}_{17}^{[s,t]} &= \begin{pmatrix} \frac{\Phi(t)}{\Phi(s)} & 0 \\ \frac{\Phi(t)}{\psi(s)}(g(t) - g(s)) & \frac{\psi(t)}{\psi(s)} \end{pmatrix}. \end{aligned}$$



$$\begin{aligned}
\mathcal{M}_{18}^{[s,t]} &= \begin{cases} \begin{pmatrix} 1 & 0 \\ \frac{h(t)-h(s)}{\psi(s)} & \frac{\psi(t)}{\psi(s)} \end{pmatrix}, & \text{if } s \leq t < a; \\ \begin{pmatrix} 0 & 0 \\ \frac{h(t)}{\psi(s)} & \frac{\psi(t)}{\psi(s)} \end{pmatrix}, & \text{if } t \geq a \end{cases}; \\
\mathcal{M}_{19}^{[s,t]} &= \begin{cases} \begin{pmatrix} 0 & 0 \\ h(t) & 1 \end{pmatrix}, & \text{if } s \leq t < b; \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } t \geq b \end{cases}; \\
\mathcal{M}_{20}^{[s,t]} &= \begin{cases} \begin{pmatrix} \frac{\Phi(t)}{\Phi(s)} & 0 \\ \Phi(t)(v(t)-v(s)) & 1 \end{pmatrix}, & \text{if } s \leq t < b; \\ \begin{pmatrix} \frac{\Phi(t)}{\Phi(s)} & 0 \\ \Phi(t)w(s) & 0 \end{pmatrix}, & \text{if } t \geq b \end{cases}; \\
\mathcal{M}_{21}^{[s,t]} &= \begin{cases} \begin{pmatrix} 1 & 0 \\ v(t)-v(s) & 1 \end{pmatrix}, & \text{if } s \leq t < \min\{a, b\}; \\ \begin{pmatrix} 1 & 0 \\ v(s) & 0 \end{pmatrix}, & \text{if } b \leq t < a, a > b; \\ \begin{pmatrix} 0 & 0 \\ v(t) & 1 \end{pmatrix}, & \text{if } a \leq t < b, a < b; \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & \text{if } t \geq \max\{a, b\} \end{cases};
\end{aligned}$$

Hence, in this case we have eight new CEAs:  $E_i^{[s,t]}$ ,  $0 \leq s \leq t$  which correspond to  $\mathcal{M}_i^{[s,t]}$ ,  $i = 14, \dots, 21$  listed above.

**Case 6.**  $a_{22}^{[s,t]} \equiv 0$ , (the case  $a_{11}^{[s,t]} \equiv 0$  is similar) then the system (3.1) reduced to following

$$\begin{aligned}
a_{11}^{[s,t]} &= a_{11}^{[s,\tau]} a_{11}^{[\tau,t]} + a_{12}^{[s,\tau]} a_{21}^{[\tau,t]}, \\
a_{12}^{[s,t]} &= a_{11}^{[s,\tau]} a_{12}^{[\tau,t]}, \\
a_{21}^{[s,t]} &= a_{21}^{[s,\tau]} a_{11}^{[\tau,t]}, \\
0 &= a_{21}^{[s,\tau]} a_{12}^{[\tau,t]}.
\end{aligned} \tag{3.8}$$

Multiplying the second and the third equations and using the fourth one we get  $a_{12}^{[s,t]} a_{21}^{[s,t]} = 0$ . Thus this case reduced to the Case 5, hence does not give any new CEA.

**Case 7.** Assume  $a_{11}^{[s,t]} + a_{12}^{[s,t]} = a_{21}^{[s,t]} + a_{22}^{[s,t]} = 1$ . Denote  $\alpha(s, t) = a_{11}^{[s,t]}$ ,  $\beta(s, t) = a_{21}^{[s,t]}$  then from (3.1) we get

$$\alpha(s, t) = \alpha(s, \tau) \alpha(\tau, t) + (1 - \alpha(s, t)) \beta(\tau, t);$$

$$\beta(s, t) = \beta(s, \tau)\alpha(\tau, t) + (1 - \beta(s, \tau))\beta(\tau, t).$$

Putting

$$(3.9) \quad \gamma(s, t) = \alpha(s, t) + \beta(s, t), \quad \delta(s, t) = \alpha(s, t) - \beta(s, t),$$

we obtain

$$(3.10) \quad \begin{aligned} \gamma(s, t) &= \gamma(s, \tau)\delta(\tau, t) + \gamma(\tau, t) - \delta(\tau, t); \\ \delta(s, t) &= \delta(s, \tau)\delta(\tau, t). \end{aligned}$$

The second equation of (3.10) has the following solutions:

- 1)  $\delta(s, t) \equiv 0$ ;
- 2)  $\delta(s, t) = \frac{\theta(t)}{\theta(s)}$ , with  $\theta(t) \neq 0$ ;
- 3)

$$\delta(s, t) = \begin{cases} 1, & \text{if } s \leq t < a, \\ 0, & \text{if } t \geq a. \end{cases} \quad \text{where } a > 0.$$

Substituting these solutions into the first equation of (3.10) we get the following

- 1')  $\gamma(s, t) = f(t)$ , where  $f$  is an arbitrary function;
- 2') For  $\delta(s, t) = \frac{\theta(t)}{\theta(s)}$  we get

$$(3.11) \quad \tilde{\gamma}(s, t) = \tilde{\gamma}(s, \tau) + \tilde{\gamma}(\tau, t) - \frac{1}{\theta(\tau)},$$

where  $\tilde{\gamma}(s, t) = \frac{\gamma(s, t)}{\theta(t)}$ . We shall find a solution of the equation (3.11) which has the form

$$\tilde{\gamma}(s, t) = \lambda \cdot u(t) - \mu \cdot u(s), \quad \lambda, \mu \in \mathbb{R}, \lambda \neq \mu.$$

We do not know another kind of solutions of the equation (3.11).

Substituting this function into equation (3.11) we get  $u(t) = \frac{1}{(\lambda - \mu)\theta(t)}$ . Consequently, we obtain

$$\gamma(s, t) = \frac{\lambda}{\lambda - \mu} - \frac{\mu\theta(t)}{(\lambda - \mu)\theta(s)}.$$

- 3') In case of 3) we get the following equation

$$\gamma(s, t) = \begin{cases} \gamma(s, \tau) + \gamma(\tau, t) - 1, & \text{if } s \leq t < a, \\ \gamma(\tau, t), & \text{if } t \geq a. \end{cases}$$

This equation has the following solution

$$\gamma(s, t) = \begin{cases} 1, & \text{if } s \leq t < a, \\ g(t), & \text{if } t \geq a, \end{cases}$$

where  $g(t)$  is an arbitrary function.

Using the obtained solutions  $\gamma$  and  $\delta$  by (3.9) we get the following matrices

$$\mathcal{M}_{22}^{[s, t]} = \begin{pmatrix} f(t) & 1 - f(t) \\ f(t) & 1 - f(t) \end{pmatrix};$$

$$\mathcal{M}_{23}^{[s,t]}(\lambda, \mu) = \begin{pmatrix} 1 - \frac{\lambda-2\mu}{2(\lambda-\mu)} \left(1 - \frac{\theta(t)}{\theta(s)}\right) & \frac{\lambda-2\mu}{2(\lambda-\mu)} \left(1 - \frac{\theta(t)}{\theta(s)}\right) \\ \frac{\lambda}{2(\lambda-\mu)} \left(1 - \frac{\theta(t)}{\theta(s)}\right) & 1 - \frac{\lambda}{2(\lambda-\mu)} \left(1 - \frac{\theta(t)}{\theta(s)}\right) \end{pmatrix};$$

$$\mathcal{M}_{24}^{[s,t]} = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & \text{if } s \leq t < a; \\ \begin{pmatrix} g(t) & 1 - g(t) \\ g(t) & 1 - g(t) \end{pmatrix}, & \text{if } t \geq a \end{cases}.$$

In this case we have three new CEAs:  $E_i^{[s,t]}$ ,  $0 \leq s \leq t$  which correspond to  $\mathcal{M}_i^{[s,t]}$ ,  $i = 22, 23, 24$  listed above.

#### 4. BARIC PROPERTY TRANSITION

In [2] a notion of property transition for CEAs is defined. We recall the definitions:

**Definition 4.1.** Assume a CEA,  $E^{[s,t]}$ , has a property, say  $P$ , at pair of times  $(s_0, t_0)$ ; one says that the CEA has  $P$  property transition if there is a pair  $(s, t) \neq (s_0, t_0)$  at which the CEA has no the property  $P$ .

Denote

$$\mathcal{T} = \{(s, t) : 0 \leq s \leq t\};$$

$$\mathcal{T}_P = \{(s, t) \in \mathcal{T} : E^{[s,t]} \text{ has property } P\};$$

$$\mathcal{T}_P^0 = \mathcal{T} \setminus \mathcal{T}_P = \{(s, t) \in \mathcal{T} : E^{[s,t]} \text{ has no property } P\}.$$

The sets have the following meaning

$\mathcal{T}_P$ -the duration of the property  $P$ ;

$\mathcal{T}_P^0$ -the lost duration of the property  $P$ ;

The partition  $\{\mathcal{T}_P, \mathcal{T}_P^0\}$  of the set  $\mathcal{T}$  is called  $P$  property diagram.

For example, if  $P$  =commutativity then since any evolution algebra is commutative, we conclude that any CEA has not commutativity property transition.

Since an evolution algebra is not a baric algebra, in general, using Theorem 2.6 we can give baric property diagram. Let us do this for the above given chains  $E_i^{[s,t]}$ ,  $i = 0, \dots, 24$ .

Denote by  $\mathcal{T}_b^{(i)}$  the baric property duration of the CEA  $E_i^{[s,t]}$ ,  $i = 0, \dots, 24$ .

**Theorem 4.2.** (i) (There is no non-baric property transition) *The algebras  $E_i^{[s,t]}$ ,  $i = 0, 1, 2, 3, 6, 10, 11, 14, 22$  are not baric for any time  $(s, t) \in \mathcal{T}$ ;*

(ii) (There is no baric property transition) *The algebras  $E_i^{[s,t]}$ ,  $i = 16, 17, 18$  and  $E_{23}^{[s,t]}(0, \mu)$ ,  $E_{23}^{[s,t]}(2\mu, \mu)$ ,  $\mu \neq 0$  are baric for any time  $(s, t) \in \mathcal{T}$ ;*

- (iii) (There is baric property transition) *The CEAs  $E_i^{[s,t]}$ ,  $i = 4, 5, 7, 8, 9, 12, 13, 15, 19, 20, 21, 24$  and  $E_{23}^{[s,t]}(\lambda, \mu)$ , with  $\lambda \notin \{0, \mu, 2\mu\}$  have baric property transition with baric property duration sets as the following*

$$\mathcal{T}_b^{(4)} = \left\{ (s, t) \in \mathcal{T} : \frac{\Phi(s)}{\Psi(s)} = \frac{\Phi(t)}{\Psi(t)} \right\}; \quad \mathcal{T}_b^{(5)} = \{(s, t) \in \mathcal{T} : s \leq t < b, \Phi(s) = \Phi(t)\};$$

$$\mathcal{T}_b^{(7)} = \{(s, t) \in \mathcal{T} : s \leq t < a, \Psi(s) = \Psi(t)\}; \quad \mathcal{T}_b^{(8)} = \{(s, t) \in \mathcal{T} : s \leq t < \min\{a, b\}\};$$

$$\mathcal{T}_b^{(9)} = \{(s, t) \in \mathcal{T} : t = s + \pi k, k \in \mathbb{Z}\}; \quad \mathcal{T}_b^{(12)} = \left\{ (s, t) \in \mathcal{T} : g(s) = \pm \frac{1}{h(s)} \right\};$$

$$\mathcal{T}_b^{(13)} = \{(s, t) \in \mathcal{T} : s \leq t < a, \psi(s) = \pm 1\};$$

$$\mathcal{T}_b^{(15)} = \{(s, t) \in \mathcal{T} : s \leq t < a, \psi(s) = 0\}; \quad \mathcal{T}_b^{(19)} = \{(s, t) \in \mathcal{T} : s \leq t < a\};$$

$$\mathcal{T}_b^{(20)} = \{(s, t) \in \mathcal{T} : s \leq t < b\} \cup \{(s, t) \in \mathcal{T} : t \geq b, w(s) = 0\};$$

$$\mathcal{T}_b^{(21)} = \{(s, t) \in \mathcal{T} : s \leq t < \max\{a, b\}\};$$

$$\mathcal{T}_b^{(23)}(\lambda, \mu) = \{(s, t) \in \mathcal{T} : \theta(t) = \theta(s)\}, \lambda \neq 0, \mu, 2\mu.$$

$$\mathcal{T}_b^{(24)} = \{(s, t) \in \mathcal{T} : s \leq t < a\}.$$

*Proof.* By Theorem 2.6 a two-dimensional evolution algebra  $E^{[s,t]}$  is baric if and only if  $a_{11}^{[s,t]} \neq 0$ ,  $a_{21}^{[s,t]} = 0$  or  $a_{22}^{[s,t]} \neq 0$ ,  $a_{12}^{[s,t]} = 0$ . The assertions of Theorem are results of the detailed checking of these conditions.  $\square$

Note that sets  $\mathcal{T}_b^{(i)}$ ,  $i = 8, 9, 19, 21, 24$  do not depend on any parameter function. But  $\mathcal{T}_b^{(i)}$ ,  $i = 4, 5, 7, 12, 13, 15, 20, 23$  depend on some parameter functions and can be controlled by choosing the corresponding parameter functions  $\Phi, \Psi, g, h, \psi, w$ . These functions are called *baric property controllers* of the CEAs. Because, they really control the baric duration set, for example, if some of them is a strong monotone function then the duration is “minimal”, i.e. the line  $s = t$ , but if a function is a constant function then the baric duration set is “maximal”, i.e. it is  $\mathcal{T}$ . Since these functions are arbitrary functions, we have a rich class of controller functions, therefore we have a “powerful” control on the property to be baric.

Now we shall compute the Lebesgue measure  $\nu$  of the sets  $\mathcal{T}_b^{(i)}$ . It is easy to see that

$$\nu(\mathcal{T}_b^{(8)}) = \frac{1}{2}(\min\{a, b\})^2; \quad \nu(\mathcal{T}_b^{(9)}) = 0; \quad \nu(\mathcal{T}_b^{(19)}) = \frac{1}{2}a^2;$$

$$\nu(\mathcal{T}_b^{(20)}) \geq \frac{1}{2}b^2; \quad \nu(\mathcal{T}_b^{(21)}) = \frac{1}{2}(\max\{a, b\})^2; \quad \nu(\mathcal{T}_b^{(24)}) = \frac{1}{2}a^2.$$

The Lebesgue measure of sets  $\mathcal{T}_b^{(i)}$ ,  $i = 4, 5, 7, 12, 13, 15, 20, 23$  depend on the corresponding controller functions.

For a given functions  $g, h, \psi$  one can easily compute  $\nu(\mathcal{T}_b^{(i)})$ ,  $i = 12, 13, 15$ . For example, if  $g, h, \psi$  are elementary functions which do not have "constant parts" in their graphs then  $\nu(\mathcal{T}_b^{(i)}) = 0$ ,  $i = 12, 13, 15$ .

**Definition 4.3.** [2] A function  $\theta$  defined on  $\mathbb{R}$  is called a function of countable variation if it has the following properties:

1. it is continuous except at most on a countable set, it has only jump-type discontinuities;
2. it has at most a countable set of singular (extremum) points.

The following theorem gives a characteristics of the baric property duration set.

**Theorem 4.4.** If the controller function of  $\mathcal{T}_b^{(i)}$ ,  $i = 4, 5, 7, 23$  (i.e.  $\frac{\Phi(t)}{\Psi(t)}$  for  $i = 4$ ;  $\Phi(t)$  for  $i = 5$ ;  $\Psi(t)$  for  $i = 7$ ;  $\theta(t)$  for  $i = 23$ ) is a function of countable variation, then the baric duration set  $\mathcal{T}_b^{(i)}$  has zero Lebesgue measure, that is the corresponding CEA is not baric almost surely.

*Proof.* Is similar to the proof of Theorem 4.8. of [2].  $\square$

## 5. ABSOLUTE NILPOTENT ELEMENTS TRANSITION.

Recall that the element  $x$  of an algebra  $A$  is called an *absolute nilpotent* if  $x^2 = 0$ .

Let  $E = \mathbb{R}^n$  be an evolution algebra over the field  $\mathbb{R}$  with structural constant coefficients matrix  $\mathcal{M} = (a_{ij})$ , then for arbitrary  $x = \sum_i x_i e_i$  and  $y = \sum_i y_i e_i \in \mathbb{R}^n$  we have

$$xy = \sum_j \left( \sum_i a_{ij} x_i y_i \right) e_j, \quad x^2 = \sum_j \left( \sum_i a_{ij} x_i^2 \right) e_j.$$

For a  $n$ -dimensional evolution algebra  $\mathbb{R}^n$  consider operator  $V: \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $x \mapsto V(x) = x'$  defined as

$$(5.1) \quad x'_j = \sum_{i=1}^n a_{ij} x_i^2, \quad j = 1, \dots, n.$$

This operator is called *evolution operator* [6].

We have  $V(x) = x^2$ , hence the equation  $V(x) = x^2 = 0$  is given by the following system

$$(5.2) \quad \sum_i a_{ij} x_i^2 = 0, \quad j = 1, \dots, n.$$

In this section we shall solve the system (5.2) for  $E_i^{[s,t]}$ ,  $i = 0, \dots, 24$ .

For a CEA  $E_i^{[s,t]}$  with matrix  $\mathcal{M}_i^{[s,t]}$  denote

$$\mathcal{T}_{nil}^{(i)} = \{(s, t) \in \mathcal{T} : E_i^{[s,t]} \text{ has unique absolute nilpotent}\}, \quad \mathcal{T}_{nil}^0 = \mathcal{T} \setminus \mathcal{T}_{nil}.$$

The following theorem gives an answer on problem of existence of “uniqueness of absolute nilpotent element” property transition.

**Theorem 5.1.** (1) *The CEAs  $E_i^{[s,t]}$ ,  $i = 3, 4, 5, 9, 10, 17, 22, 23, 24$  have unique absolute nilpotent element  $(0, 0)$  for any time  $(s, t) \in \mathcal{T}$ .*  
 (2) *There CEAs  $E_i^{[s,t]}$ ,  $i = 0, 1, 2, 16, 19$  have infinitely many of absolute nilpotent elements for any time  $(s, t) \in \mathcal{T}$ .*  
 (3) *The CEAs  $E_i^{[s,t]}$ ,  $i = 6, 7, 8, 11, 12, 13, 14, 15, 18, 20, 21$  have “uniqueness of absolute nilpotent element” property transition with the property duration sets as the following*

$$\mathcal{T}_{nil}^{(i)} = \{(s, t) \in \mathcal{T} : t < a\}, \quad a > 0, \quad i = 6, 7, 8, 11, 18.$$

$$\mathcal{T}_{nil}^{(12)} = \left\{ (s, t) \in \mathcal{T} : g^2(t) \leq \frac{1}{h^2(s)} \right\};$$

$$\mathcal{T}_{nil}^{(13)} = \{(s, t) \in \mathcal{T} : s \leq t < a, \psi^2(s) \leq 1\}; \quad \mathcal{T}_{nil}^{(14)} = \{(s, t) \in \mathcal{T} : \Phi(s)\psi(s) > 0\};$$

$$\mathcal{T}_{nil}^{(15)} = \{(s, t) \in \mathcal{T} : s \leq t < a, \psi(s) > 0\};$$

$$\mathcal{T}_{nil}^{(20)} = \{(s, t) \in \mathcal{T} : s \leq t < b\} \cup \left\{ (s, t) \in \mathcal{T} : t \geq b, \frac{w(s)}{\Phi(s)} > 0 \right\};$$

$$\mathcal{T}_{nil}^{(21)} = \{(s, t) \in \mathcal{T} : s \leq t < \min\{a, b\}\} \cup \{(s, t) \in \mathcal{T} : b \leq t < a, b < a, v(s) > 0\}.$$

*Proof.* The proof consists the simple analysis of the solutions of the system (5.2) for each  $E_i^{[s,t]}$ ,  $i = 0, \dots, 24$ .  $\square$

## 6. IDEMPOTENT ELEMENTS TRANSITION

A element  $x$  of an algebra  $\mathcal{A}$  is called *idempotent* if  $x^2 = x$ ; such points of an evolution algebra are especially important, because they are the fixed points (i.e.  $V(x) = x$ ) of the evolution operator  $V$ , (5.1). We denote by  $\mathcal{Id}(E)$  the set of idempotent elements of an algebra  $E$ . Using (5.1) the equation  $x^2 = x$  can be written as

$$(6.1) \quad x_j = \sum_{i=1}^n a_{ij} x_i^2, \quad j = 1, \dots, n.$$

The general analysis of the solutions of the system (6.1) is very difficult. We shall solve this problem for the CEAs  $E_i^{[s,t]}$ ,  $i = 0, \dots, 24$ .

The following theorem gives the time-dynamics of the idempotent elements for algebras  $E_i^{[s,t]}$ ,  $i = 0, \dots, 24$ .

**Theorem 6.1.** (1) *The algebras  $E_i^{[s,t]}$ ,  $i = 0, 1, 2$ , have unique idempotent  $(0, 0)$  any time  $(s, t) \in \mathcal{T}$ .*  
 (2) *The algebras  $E_i^{[s,t]}$ ,  $i = 3, 10, 12, 14, 16, 22$  have two idempotents  $(0, 0)$ ,  $(x_i(s, t), y_i(s, t))$  any time  $(s, t) \in \mathcal{T}$ . Moreover explicit formula of each  $x_i(s, t)$  and  $y_i(s, t)$  can be given.*

(3) *We have*

$$\mathcal{Id}\left(E_4^{[s,t]}\right) = \begin{cases} \{0, z_1, z_2, z_3\}, & \text{if } (s, t) \in \left\{(s, t) \in \mathcal{T} : \frac{\Phi(t)}{\Phi(s)} = \frac{\Psi(t)}{\Psi(s)}\right\} \\ \{0, z_3\}, & \text{if } (s, t) \in \left\{\frac{\Phi(t)}{\Phi(s)} \neq \frac{\Psi(t)}{\Psi(s)}, D(s, t) < 0\right\} \\ \{0, z_3, (x_*, y_*)\}, & \text{if } (s, t) \in \left\{\frac{\Phi(t)}{\Phi(s)} \neq \frac{\Psi(t)}{\Psi(s)}, D(s, t) = 0\right\} \\ \{0, z_3, (x_{\pm}, y_{\pm})\}, & \text{if } (s, t) \in \left\{\frac{\Phi(t)}{\Phi(s)} \neq \frac{\Psi(t)}{\Psi(s)}, D(s, t) > 0\right\}, \end{cases}$$

where  $0 = (0, 0)$ ,  $z_1 = (0, \frac{\Phi(t)}{\Phi(s)})$ ,  $z_2 = (\frac{\Phi(t)}{\Phi(s)}, 0)$ ,  $z_3 = (\frac{\Phi(t)}{\Phi(s)}, \frac{\Phi(t)}{\Phi(s)})$ ,  $D(s, t) = \frac{\Phi(t)}{\Phi(s)} \left(2\frac{\Psi(t)}{\Psi(s)} - \frac{\Phi(t)}{\Phi(s)}\right)$ . The explicit formulas of  $x_*$ ,  $y_*$ ,  $x_{\pm}$  and  $y_{\pm}$  are given below. The sets  $\left\{\frac{\Phi(t)}{\Phi(s)} = \frac{\Psi(t)}{\Psi(s)}\right\}$ ,  $\left\{\frac{\Phi(t)}{\Phi(s)} = 2\frac{\Psi(t)}{\Psi(s)}\right\}$  are critical (boundary) sets of the idempotent elements transition.

(4) *We have*

$$\mathcal{Id}\left(E_5^{[s,t]}\right) = \begin{cases} \{0, z_1, z_2, z_3\}, & \text{if } (s, t) \in \{(s, t) \in \mathcal{T} : s \leq t < b, \Phi(t) = \Phi(s)\} \\ \{0, z_3\}, & \text{if } (s, t) \in \{s \leq t < b, \Phi(t) \neq \Phi(s), D(s, t) < 0\} \\ \{0, z_3, (x_*, y_*)\}, & \text{if } (s, t) \in \{s \leq t < b, \Phi(t) \neq \Phi(s), D(s, t) = 0\} \\ \{0, z_3, (x_{\pm}, y_{\pm})\}, & \text{if } (s, t) \in \{s \leq t < b, \Phi(t) \neq \Phi(s), D(s, t) > 0\}, \\ \{0, z_3\}, & \text{if } t \geq b. \end{cases}$$

where  $z_i$  as in (3),  $D(s, t) = \frac{\Phi(t)}{\Phi(s)} \left(2 - \frac{\Phi(t)}{\Phi(s)}\right)$ .

(5) *Algebras  $E_i^{[s,t]}$ ,  $i = 6, 11, 13, 15, 19$  have two idempotent elements for any time  $(s, t)$  with  $s \leq t < a$  and a unique idempotent for time  $(s, t)$  with  $t \geq a$ . The critical line of the transition is  $t = a$ .*

(6) *We have*

$$\mathcal{Id}\left(E_7^{[s,t]}\right) = \begin{cases} \{0, z_1, z_2, z_3\}, & \text{if } (s, t) \in \{(s, t) \in \mathcal{T} : s \leq t < a, \Psi(t) = \Psi(s)\} \\ \{0, z_3\}, & \text{if } (s, t) \in \{s \leq t < a, \Psi(t) \neq \Psi(s), d(s, t) < 0\}, \\ \{0, z_3, (x_*, y_*)\}, & \text{if } (s, t) \in \{s \leq t < a, \Psi(t) \neq \Psi(s), d(s, t) = 0\}, \\ \{0, z_3, (x_{\pm}, y_{\pm})\}, & \text{if } (s, t) \in \{s \leq t < a, \Psi(t) \neq \Psi(s), d(s, t) > 0\}, \\ 0, & \text{if } t \geq a, \end{cases}$$

where  $d(s, t) = \frac{2\Psi(t)}{\Psi(s)} - 1$ . The critical sets are  $t = a$ ,  $\Psi(t) = \Psi(s)$ ,  $\Psi(s) = 2\Psi(t)$ .

(7) *For  $a \leq b$  we have*

$$\mathcal{Id}\left(E_8^{[s,t]}\right) = \begin{cases} \{(0, 0), (0, 1), (1, 0), (1, 1)\}, & \text{if } (s, t) \in \{(s, t) \in \mathcal{T} : s \leq t < a\} \\ (0, 0), & \text{if } t \geq a; \end{cases}$$

For  $a > b$  we have

$$\mathcal{Id}\left(E_8^{[s,t]}\right) = \begin{cases} \{(0,0), (0,1), (1,0), (1,1)\}, & \text{if } (s,t) \in \{(s,t) \in \mathcal{T} : s \leq t < b\} \\ \{(0,0), (1,1)\}, & \text{if } (s,t) \in \{(s,t) \in \mathcal{T} : b \leq t < a\} \\ (0,0), & \text{if } t \geq a; \end{cases}$$

The lines  $t = a$  and  $t = b$  are critical for the transition.

- (8) The algebra  $E_9^{[s,t]}$ , has three idempotent elements  $(0,0), (1,0), (0,1)$  for any time  $(s,t)$  with  $t = s + 2\pi n$ ; has three idempotent elements  $(0,0), (-1,0), (0,-1)$  for any time  $(s,t)$  with  $t = s + (2n+1)\pi$  and at least one idempotent for time  $(s,t)$  with  $t \neq s + \pi n, n \in \mathbb{Z}$ .
- (9) We have

$$\mathcal{Id}\left(E_{17}^{[s,t]}\right) = \begin{cases} \{(0,0), z_2\}, & \text{if } D(s,t) < 0 \\ \{(0,0), z_2, (\frac{\Phi(s)}{2\Phi(t)}, \frac{\Psi(s)}{\Psi(t)})\}, & \text{if } D(s,t) = 0 \\ \{(0,0), z_2, (x_{\pm}, y_{\pm})\}, & \text{if } D(s,t) > 0, \end{cases}$$

where  $D(s,t) = \frac{4\Phi^2(t)\Psi(s)}{\Phi(s)\Psi^2(t)}(g(t) - g(s)) - 1$ .

- (10) We have

$$\mathcal{Id}\left(E_{18}^{[s,t]}\right) = \begin{cases} \{(0,0), (1,0)\}, & \text{if } s \leq t < a, D(s,t) < 0, \\ \{(0,0), (1,0), (\frac{1}{2}, \frac{\Psi(s)}{\Psi(t)})\}, & \text{if } s \leq t < a, D(s,t) = 0, \\ \{(0,0), (1,0), (x_{\pm}, \frac{\Psi(s)}{\Psi(t)})\}, & \text{if } s \leq t < a, D(s,t) > 0, \\ \{(0,0), (\frac{h(t)\Psi(s)}{\Psi^2(t)}, \frac{\Psi(s)}{\Psi(t)})\}, & \text{if } t \geq a, \end{cases}$$

where  $D(s,t) = 1 - \frac{4\Psi(s)(h(t)-h(s))}{\Psi^2(t)}$ .

- (11) We have

$$\mathcal{Id}\left(E_{20}^{[s,t]}\right) = \begin{cases} \{(0,0), z_2\}, & \text{if } s \leq t < b, D(s,t) < 0, \\ \{(0,0), z_2, (\frac{\Phi(s)}{2\Phi(t)}, 1)\}, & \text{if } s \leq t < b, D(s,t) = 0, \\ \{(0,0), z_2, (x_{\pm}, 1)\}, & \text{if } s \leq t < a, D(s,t) > 0, \\ \{(0,0), z_2\} & \text{if } t \geq b, \end{cases}$$

where  $D(s,t) = 1 - \frac{4\Phi^2(t)(v(t)-v(s))}{\Phi(s)}$ .



(12) *We have*

$$\mathcal{Id}\left(E_{21}^{[s,t]}\right) = \begin{cases} \{(0,0), (1,0)\}, & \text{if } s \leq t < \min\{a, b\}, D(s, t) < 0, \\ \{(0,0), (1,0), (\frac{1}{2}, 1)\}, & \text{if } s \leq t < \min\{a, b\}, D(s, t) = 0, \\ \{(0,0), (1,0), (x_{\pm}, 1)\}, & \text{if } s \leq t < \min\{a, b\}, D(s, t) > 0, \\ \{(0,0), (1,0)\}, & \text{if } b < a, b \leq t < a, \\ \{(0,0), (v(t), 1)\}, & \text{if } b > a, a \leq t < b, \\ (0,0), & \text{if } t \geq \max\{a, b\}, \end{cases}$$

where  $D(s, t) = 1 - 4(v(t) - v(s))$ .

(13) *We have*

$$\mathcal{Id}\left(E_{23}^{[s,t]}(0, \mu)\right) = \begin{cases} \{(0,0), (0,1)\}, & \text{if } D(s, t) < 0 \\ \{(0,0), (0,1), (\frac{\theta(s)}{\theta(t)}, \frac{1}{2})\}, & \text{if } D(s, t) = 0 \\ \{(0,0), (0,1), (\frac{\theta(s)}{\theta(t)}, y_{\pm})\}, & \text{if } D(s, t) > 0; \end{cases}$$

$$\mathcal{Id}\left(E_{23}^{[s,t]}(2\mu, \mu)\right) = \begin{cases} \{(0,0), (1,0)\}, & \text{if } D(s, t) < 0 \\ \{(0,0), (1,0), (\frac{1}{2}, \frac{\theta(s)}{\theta(t)})\}, & \text{if } D(s, t) = 0 \\ \{(0,0), (1,0), (x_{\pm}, \frac{\theta(s)}{\theta(t)})\}, & \text{if } D(s, t) > 0, \end{cases}$$

where  $D(s, t) = 1 - \frac{4\theta(s)}{\theta(t)} \left(\frac{\theta(s)}{\theta(t)} - 1\right)$ .

(14) *We have*

$$\mathcal{Id}\left(E_{24}^{[s,t]}\right) = \begin{cases} \{(0,0), (0,1), (1,0), (1,1)\}, & \text{if } (s, t) \in \{(s, t) \in \mathcal{T} : s \leq t < a\} \\ \{(0,0), \left(\frac{g(t)}{(1-g(t))^2 + g^2(t)}, \frac{1-g(t)}{(1-g(t))^2 + g^2(t)}\right)\}, & \text{if } t \geq a. \end{cases}$$

*Proof.* The proof contains detailed analysis of solutions of the system (6.1) for each  $E_i^{[s,t]}$ . We shall give here proof of the assertion (3) which is more substantial. In case of  $E_4^{[s,t]}$  the system (6.1) has the following form

$$(6.2) \quad \begin{cases} 2x = \left(\frac{\Phi(t)}{\Phi(s)} + \frac{\Psi(t)}{\Psi(s)}\right) x^2 + \left(\frac{\Phi(t)}{\Phi(s)} - \frac{\Psi(t)}{\Psi(s)}\right) y^2; \\ 2y = \left(\frac{\Phi(t)}{\Phi(s)} - \frac{\Psi(t)}{\Psi(s)}\right) x^2 + \left(\frac{\Phi(t)}{\Phi(s)} + \frac{\Psi(t)}{\Psi(s)}\right) y^2. \end{cases}$$

*Case  $\Phi(t) \equiv \Psi(t)$ .* It is easy to see that in this case the system (6.2) has only four solutions  $0 = (0, 0)$  and

$$z_1 = z_1(s, t) = \left(0, \frac{\Phi(s)}{\Phi(t)}\right), \quad z_2 = z_2(s, t) = \left(\frac{\Phi(s)}{\Phi(t)}, 0\right), \quad z_3 = z_3(s, t) = \left(\frac{\Phi(s)}{\Phi(t)}, \frac{\Phi(s)}{\Phi(t)}\right).$$

*Case  $\Phi(t) \neq \Psi(t)$ .* In this case the solutions  $0$  and  $z_3$  still exist.

*Subcase  $x = 0$ .* For  $x = 0$  we have only solution  $(0, 0)$  if  $\frac{\Phi(t)}{\Phi(s)} \neq \frac{\Psi(t)}{\Psi(s)}$  and there are two solutions  $(0, 0)$  and  $z_1$  if  $\frac{\Phi(t)}{\Phi(s)} = \frac{\Psi(t)}{\Psi(s)}$ .

*Subcase  $y = 0$ .* For  $y = 0$  we have only solution  $(0, 0)$  if  $\frac{\Phi(t)}{\Phi(s)} \neq \frac{\Psi(t)}{\Psi(s)}$  and there are two solutions  $(0, 0)$  and  $z_2$  if  $\frac{\Phi(t)}{\Phi(s)} = \frac{\Psi(t)}{\Psi(s)}$ .

*Subcase  $xy \neq 0$ .* Set  $u = \frac{x}{y}$ . From system (6.2) we get

$$(6.3) \quad (u - 1) \left( \left( \frac{\Phi(t)}{\Phi(s)} - \frac{\Psi(t)}{\Psi(s)} \right) u^2 - 2 \frac{\Psi(t)}{\Psi(s)} u + \left( \frac{\Phi(t)}{\Phi(s)} - \frac{\Psi(t)}{\Psi(s)} \right) \right) = 0.$$

This equation has unique solution  $u = 1$  if  $\frac{\Phi(t)}{\Phi(s)} = \frac{\Psi(t)}{\Psi(s)}$  or  $\frac{\Phi(t)}{\Phi(s)} \neq \frac{\Psi(t)}{\Psi(s)}$  and  $D(s, t) = \frac{\Phi(t)}{\Phi(s)} \left( 2 \frac{\Psi(t)}{\Psi(s)} - \frac{\Phi(t)}{\Phi(s)} \right) < 0$ . For  $\frac{\Phi(t)}{\Phi(s)} \neq \frac{\Psi(t)}{\Psi(s)}$  it has two solutions  $u = 1$  and  $u = u_*(s, t)$  if  $D(s, t) = 0$  and three solutions  $u = 1, u = u_{\pm}(s, t)$  if  $D > 0$ .

Now one can describe  $x, y$  corresponding to the solutions of (6.3). The case  $u = 1$ , i.e.  $x = y$  does not give any new solution. For  $u = u_*, u_{\pm}$  we have  $x = u_* y$  and  $x = u_{\pm} y$ , substituting these in the second equation of (6.2) after simple calculations we get the following non-zero solutions to (6.2):

$$x_* = \frac{\Phi(s)}{\Phi(t)}, \quad y_* = \frac{\Psi(s)}{\Psi(t)} - \frac{\Phi(s)}{\Phi(t)};$$

$$x_{\pm} = \frac{1 \pm \frac{\Psi(s)}{\Psi(t)} \sqrt{D(s, t)}}{\frac{\Phi(t)}{\Phi(s)} \pm \sqrt{D(s, t)}}, \quad y_{\pm} = \frac{\frac{\Phi(t)}{\Phi(s)} - \frac{\Psi(t)}{\Psi(s)}}{\frac{\Psi(t)}{\Psi(s)} \left( \frac{\Phi(t)}{\Phi(s)} \pm \sqrt{D(s, t)} \right)}.$$

Note that  $x_{\pm}, y_{\pm}$  are well defined for any  $(s, t)$  with  $\frac{\Phi(t)}{\Phi(s)} \neq \frac{\Psi(t)}{\Psi(s)}$ . Thus the critical (boundary) times of the transition of idempotent elements are points  $(s, t)$  which satisfy  $\frac{\Phi(t)}{\Phi(s)} = \frac{\Psi(t)}{\Psi(s)}$  or  $\frac{\Phi(t)}{\Phi(s)} = \frac{2\Psi(t)}{\Psi(s)}$ .

Proofs of the assertions (i),  $i = 1, 2, 4 - 14$  are similar analysis of the system (6.1) for each  $E_i^{[s, t]}$ .  $\square$

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